An inverse scattering problem for the Schrödinger equation in a semiclassical process.

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Abstract

We study an inverse scattering problem for a pair of Hamiltonians $(H(h), H_0(h))$ on $L^2(\mathbb{R}^n)$, where $H_0(h) = -h^2\Delta$ and $H(h) = H_0(h) + V$, V is a short-range potential with a regular behaviour at infinity and h is the semiclassical parameter. We show that, in dimension $n \geq 3$, the knowledge of the scattering operators $S(h)$, $h \in]0,1]$, up to $O(h^\infty)$ in $\mathcal{B}(L^2(\mathbb{R}^n))$, and which are localized near a fixed energy $\lambda > 0$, determine the potential V at infinity.

1 Introduction.

This work is a continuation of [10]. In the present paper, we study an inverse scattering problem for the pair of Hamiltonians $(H(h), H_0(h))$ on $L^2(\mathbb{R}^n)$, $n \geq 2$, where the free operator is $H_0(h) = -h^2\Delta$, $h \in]0,1]$ is the semiclassical parameter, and the perturbed Hamiltonian is given by :

$$H(h) = H_0(h) + V. \tag{1.1}$$

We assume that V is a real-valued smooth short-range potential satisfying :

$$(H_1) \qquad \forall \alpha \in \mathbb{N}^n \ \ , \ \ \exists\, C_\alpha > 0 \ \ , \ \ |\, \partial_x^\alpha V(x) \,| \leq \ C_\alpha \ <x>^{-\rho-|\alpha|} \ , \ \rho > 1.$$

Under the hypothesis (H_1), it is well-known that the wave operators :

$$W^{\pm}(h) \;=\; s - \lim_{t \to \pm\infty} \; e^{itH(h)} e^{-itH_0(h)}, \tag{1.2}$$

exist and are complete, i.e $Ran\ W^{\pm}(h) = \mathcal{H}_{ac}(H) =$ subspace of absolute continuity of $H(h)$.

Let $S(h)$ be the scattering operator defined by :

$$S(h) = W^{+*}(h)\ W^{-}(h). \tag{1.3}$$

Since $S(h)$ commutes with $H_0(h)$, we can define the scattering matrices $S(\lambda, h)$, $\lambda > 0$, as unitary operators acting on $L^2(S^{n-1})$, where S^{n-1} is the unit sphere in $I\!R^n$. We denote by $\Phi_0(x, \lambda, \omega, h)$, $(\lambda, \omega) \in]0, +\infty[\times S^{n-1}$, the generalized eigenfunction of $H_0(h)$:

$$\Phi_0(x, \lambda, \omega, h) = e^{\frac{i}{h}\sqrt{\lambda} x \cdot \omega}. \tag{1.4}$$

The unitary mapping $\mathcal{F}_0(h)$ defined by :

$$(\mathcal{F}_0(h) f)(\lambda, \omega) = (2\pi h)^{-\frac{n}{2}}\ \lambda^{\frac{n-2}{4}} \int_{I\!R^n} \overline{\Phi_0}(x, \lambda, \omega, h)\ f(x)\ dx\ , \tag{1.5}$$

gives the spectral representation for $H_0(h)$, i.e $H_0(h)$ is transformed into the multiplication by λ in the space $L^2\ (I\!R^+\ ;\ L^2(S^{n-1}))$.

Then, the scattering matrices are defined by :

$$\mathcal{F}_0(h) S(h) f(\lambda) = S(\lambda, h)\ (\mathcal{F}_0(h) f)(\lambda), \tag{1.6}$$

and we have the Kuroda's representation formula :

$$S(\lambda, h) = 1 - 2i\pi F_0(\lambda, h) V F_0(\lambda, h)^* + 2i\pi F_0(\lambda, h) V R(\lambda + i0, h) V F_0(\lambda, h)^*\ , \tag{1.7}$$

where for $f \in L^2_s(I\!R^n)$, $s > \frac{1}{2}$, $F_0(\lambda, h) f(\omega) = (\mathcal{F}_0(h) f)(\lambda, \omega)$ and $R(\lambda + i0, h)$ is given by the well-known principle of limiting absorption.

We recall the non-trapping condition. Let $(z(t, x, \xi),\ \zeta(t, x, \xi))$ be the solution to the Hamilton equations :

$$\dot{z}(t, x, \xi) = 2\zeta(t, x, \xi) \quad , \quad \dot{\zeta}(t, x, \xi) = -\nabla V(z(t, x, \xi)), \tag{1.8}$$

with the initial data $z(0, x, \xi) = x$, $\zeta(0, x, \xi) = \xi$.

We say that the energy λ is non-trapping, if for any $R \gg 1$ large enough, there exists $T = T(R)$ such that $\mid z(t, x, \xi) \mid > R$ for $\mid t \mid > T$, when $\mid x \mid < R$ and $\lambda = \xi^2 + V(x)$.

When λ is a non-trapping energy, and if V satisfies (H_1) with $\rho > 0$, we have the following estimate when $h \to 0$, (see [12], [14]),

$$\forall s > \frac{1}{2} \quad , \quad \| < x >^{-s}\ \ R(\lambda + i0, h)\ \ < x >^{-s} \| = O\ (\frac{1}{h}). \tag{1.9}$$

The goal of this paper is to give a partial answer to the following conjecture for a class of potentials which are regular at infinity. We denote by $\mathcal{B}(\mathcal{H})$ the space of bounded operators acting on a Hilbert space $\mathcal{H}$, and by $f_+(x) = max\ (f(x), 0)$.

Conjecture.

Let V_1 and V_2 be potentials satisfying (H_1) and let $S_j(\lambda, h)$, $j = 1, 2$, be the corresponding scattering matrices at a fixed non-trapping energy $\lambda > 0$. Assume that, in the semiclassical regime $h \to 0$, $S_1(\lambda, h) = S_2(\lambda, h) + O(h^\infty)$, in $\mathcal{B}(L^2(S^{n-1}))$.
Then, the potentials are equal in the classical allowed region, i.e

$$(\lambda - V_1(x))_+ = (\lambda - V_2(x))_+. \tag{1.10}$$

According to the author, this result is not known in the general case. In [11], for potentials satisfying (H_1) with $\rho > n$, Novikov shows, without the non-trapping condition and using the $\overline{\partial}$-method, that if $S_1(\lambda, h) = S_2(\lambda, h)$, $\forall h \in]0, 1[$, then $V_1 = V_2$. It is clear that this approach is not adapted to study our conjecture in the semiclassical setting since, physically, we can not obtain any information on the potential outside the classical allowed region. To give an example, in [13], Robert and Tamura obtain, under suitable assumptions, the complete asymptotic expansion of the scattering amplitude $f(\lambda, \theta, \omega, h)$, with θ, ω fixed, $\theta \neq \omega$, and where λ is a fixed non-trapping energy. The coefficients of this expansion depend only on the values of $V(x)$ in $\{x \in I\!\!R^n \ : \ V(x) \leq \lambda\}$. It follows that if we modify the potential V outside the classical allowed region, the scattering amplitude remains unchanged up to $O(h^\infty)$.

It seems to be difficult to prove our conjecture for a fixed energy without assuming the non-trapping condition. The difficulty comes from the estimate of the resolvent $R(\lambda + i0, h)$ due to resonances converging exponentially to the real axis. To avoid the problem due to these resonances, one can average with respect to the energy λ. In [16], Yajima obtains, without the non-trapping condition and with the additional condition $\rho > max\ (1, \frac{n-1}{2})$, the asymptotic expansion of the scattering amplitude, averaging with respect to λ and θ . In [6], Michel proves for a similar result when one averages only with respect to λ, i.e. $\theta \neq \omega$ are fixed. In [7], he obtains for a fixed trapping energy, but with an additional assumption on the resonances, the asymptotic expansion of the scattering amplitude similar to the one established in the non-trapping case.

For potentials with compact support, an inverse scattering problem close to our conjecture is studied in [1]. Assuming that $\lambda > \sup_{x \in I\!\!R^n} V_+(x)$ and that the metric $g(x) = (\lambda - V(x))\ dx^2$ is conformal to the Euclidian, Alexandrova shows in [1] that the scattering amplitude in the semiclassical regime, (actually, the scattering relations), determines V uniquely. This result is obtained by comparing the inverse scattering problem for the Schrödinger equation with the problem of the wave equation with variable speed for which the result is well-known. The scattering relations for the metric $g(x)$ determine the boundary distance function uniquely, and then we determine g and therefore $V(x)$. This method can not be used for general potentials satisfying (H_1).

Now, let us explain more precisely the setting of this paper. We consider the class of potentials V which are asymptotic sums of homogenous terms at infinity, i.e. we assume that $V \in C^\infty(I\!R^n)$ and satisfies the regular behaviour at infinity :

$$(H_2) \qquad V(x) \simeq \sum_{j=1}^{\infty} V_j(x)$$

where the $V_j(x)$ are homogeneous functions of order $-\rho_j$ with $1 < \rho_1 < \rho_2 < \cdots$.

Inverse scattering at a fixed energy with regular potentials at infinity and when the Planck's constant $h = 1$ was first studied in [5] with $\rho_j = j + 1$. In [15], Weder and Yafaev study a similar problem for general degrees of homogeneity ρ_j. They show that the complete asymptotic expansion of the potential is determined by the singularities in the forward direction of the scattering amplitude. In particular, they obtain the following result :

Let V_1, V_2 be potentials satisfy the assumption (H_2) and let $S_j(\lambda)$, $j = 1, 2$, be the corresponding scattering matrices. Assume that the kernel of $S_1(\lambda) - S_2(\lambda)$ belongs to $C^\infty(S^{n-1} \times S^{n-1})$, then $V_1 - V_2$ belongs to the Schwartz class $\mathcal{S}(I\!R^n)$.

If we want to study the inverse scattering problem in the semiclassical setting with a fixed non-trapping energy and for potentials having a regular behaviour at infinity, it seems to the author that the technics developed in [15] are not well adapted. Indeed, as it was pointed in ([12], [13]), for potentials satisfying (H_1) with $\rho > n$, the forward amplitude $f(\lambda, \omega, \omega, h)$ is of order $O\ (h^{-\mu})$ with $\mu = \dfrac{(n-1)(n+1)}{2(\rho - 1)}$ and $f(\lambda, \theta, \omega, h)$ with $\theta \neq \omega$ is of order $O(1)$. Thus, the scattering amplitude has a strong peak in a neighborhood of ω.

Moreover, let us emphasize that, generically, we do not have any information on the kernel of $S_1(\lambda, h) - S_2(\lambda, h) = O(h^\infty)$. These are the reasons why we prefer to work with an arbitrary small interval of positives energies and we use the approach given in ([8], [9], [10]) which is a time-independent version of [2].

So, our goal in this paper is to recover all functions V_j from the knowledge of the scattering operator $S(h)$ localized near a fixed energy λ, up to $O(h^\infty)$ in the sense of the operator norm in $L^2(I\!R^n)$. Since we do not work with a fixed energy, but in an averaged sense, we emphasize that the non-trapping condition is not necessary.

In order to localize the scattering operator near a fixed energy $\lambda > 0$, we introduce a cut-off function $\chi \in C_0^\infty(]0, +\infty[)$, $\chi = 1$ in a neighborhood of $\lambda > 0$.

We show that we obtain the complete asymptotic expansion of the potential from the knowledge of $S(h)\chi(H_0(h))$ up to $O\ (h^\infty)$ in $\mathcal{B}(L^2(I\!R^n))$, when $h \to 0$. More precisely, we determine, for $n \geq 3$, the asymptotics expansion of the potential at infinity, by studying the asymptotics of :

$$(1.11) \qquad F(h) =< S(h)\chi(H_0(h))\Phi_{h,\omega}, \Psi_{h,\omega} > ,$$

where $<,>$ is the usual scalar product in $L^2(I\!R^n)$, and $\Phi_{h,\omega}$, $\Psi_{h,\omega}$ are suitable test functions, (see section 2). We need $n \geq 3$ in order to use the uniqueness for the Radon transform in hyperplanes.

2 Semiclassical asymptotics for the localized scattering operator.

2.1 Definition of the test functions.

First, let us define the unitary dilation operator $U(h^\delta)$, $\delta > 0$, on $L^2(I\!R^n)$ by :

$$U(h^\delta)\ \Phi(x) = h^{\frac{n\delta}{2}}\ \Phi(h^\delta x). \tag{2.1}$$

We also need an energy cut-off $\chi_0 \in C_0^\infty(I\!R^n)$ such that $\chi_0(\xi) = 1$ if $\mid \xi \mid \leq 1$, $\chi_0(\xi) = 0$ if $\mid \xi \mid \geq 2$.

For $\omega \in S^{n-1}$, we write $x \in I\!R^n$ as $x = y + t\omega$, $y \in \Pi_\omega$ = orthogonal hyperplane to ω and we consider :

$$X_\omega = \{ x = y + t\omega \in I\!R^n\ : \mid y \mid \geq 1\ \}. \tag{2.2}$$

For $\delta > \dfrac{1}{\rho_1 - 1}$ and $\epsilon < 1 + \delta$, we define :

$$\Phi_{h,\omega} = e^{\frac{i}{h}\sqrt{\lambda} x.\omega}\ U(h^\delta)\ \chi_0(h^\epsilon D)\ \Phi\ , \tag{2.3}$$

where $\Phi \in C_0^\infty(X_\omega)$ and $D = -i\nabla$, ($\Psi_{h,\omega}$ is defined in the same way with $\Psi \in C_0^\infty(X_\omega)$).

2.2 Semiclassical asymptotics for the scattering operator.

In this paper, we prove the following theorem :

Theorem 1
Let V be a potential satisfying (H_2). Then, there exists an increasing positive sequence $(\nu_k)_{k\geq 1}$, (depending only on δ), with $\nu_1 = \delta(\rho_1 - 1) - 1$ and $\lim\limits_{k\to+\infty} \nu_k = +\infty$ such that :

$$< (S(h) - 1)\ \chi(H_0(h))\ \Phi_{h,\omega}\ ,\ \Psi_{h,\omega} > \simeq \sum_{k=1}^{\infty} h^{\nu_k}\ < \Phi\ ,\ A_k(x,\omega,D)\ \Psi > , \tag{2.4}$$

when $h \to 0$, where $A_j(x,\omega,D)$ are differential operators defined in a recursive way.

Moreover, $\forall j \geq 1, \exists\ k_j \geq 1$, (with $k_1 = 1$), such that :

$$A_{k_j}(x,\omega,D) = \frac{i}{2\sqrt{\lambda}} \int_{-\infty}^{+\infty} V_j(x + t\omega)\ dt\ +\ B_j(x,\omega,D), \tag{2.5}$$

with $B_1 = 0$ and for $j \geq 2$, $B_j(x,\omega,D)$ is a differential operator only depending on the functions V_k, $1 \leq k \leq j-1$.

2.3 Proof of Theorem 1.

Step 1 :

Let us begin by an elementary lemma.

Lemma 2
$\forall h \ll 1$ *small enough, we have :*

$$\chi(H_0(h))\Phi_{h,\omega} \;=\; \Phi_{h,\omega}. \tag{2.6}$$

Proof
We easily obtain :

$$\mathcal{F}\,[\chi(H_0(h))\Phi_{h,\omega}](\xi) \;=\; h^{-\frac{n\delta}{2}}\chi((h\xi)^2)\chi_0(h^{\epsilon-\delta-1}(h\xi-\sqrt{\lambda}\omega))\mathcal{F}\Phi(h^{-\delta-1}(h\xi-\sqrt{\lambda}\omega)), \tag{2.7}$$

where $\mathcal{F}$ is the usual Fourier transform. Then, on $Supp\ \chi_0$, we have $\mid h\xi-\sqrt{\lambda}\omega \mid \leq 2h^{1+\delta-\epsilon}$. Since $\epsilon < 1+\delta$, we have for h small enough, $\chi((h\xi)^2)=1$. $\square$

Then, by Lemma 2, we obtain,

$$F(h) = < W^-(h)\Phi_{h,\omega}\ ,\ W^+(h)\Psi_{h,\omega} >, \tag{2.8}$$

and an easy calculation gives :

$$F(h) = < \Omega^-(h,\omega)\chi_0(h^\epsilon D)\Phi\ ,\ \Omega^+(h,\omega)\chi_0(h^\epsilon D)\Psi >, \tag{2.9}$$

where

$$\Omega^{\pm}(h,\omega) = s - \lim_{t\to\pm\infty}\ e^{itH(h,\omega)}e^{-itH_0(h,\omega)}, \tag{2.10}$$

with

$$H_0(h,\omega) = (D+\sqrt{\lambda}h^{-(1+\delta)}\omega)^2, \tag{2.11}$$

and

$$H(h,\omega) = H_0(h,\omega) + h^{-2(1+\delta)}V(h^{-\delta}x). \tag{2.12}$$

So, by (2.9), we have to find the asymptotics of $\Omega^{\pm}(h,\omega)\chi_0(h^\epsilon D)\Phi$. We follow the same strategy as in [10], and we only treate only the case $(+)$.

Step 2 :

As in [10], we construct, for a suitable sequence (ν_k) defined below, a modifier $J^+(h,\omega)$ as a pseudodifferential operator, (actually a differential operator close to Isozaki-Kitada's construction [4]), having the asymptotic expansion :

$$J^+(h,\omega) = \ op\left(1+\sum_{k\geq 1}\ h^{\nu_k}\ d_k^+(x,\xi,\omega)\right). \tag{2.13}$$

We denote :

$$T^+(h,\omega) = H(h,\omega)J^+(h,\omega) - J^+(h,\omega)H_0(h,\omega). \tag{2.14}$$

A direct calculation shows that the symbol of $T^+(h,\omega)$ is given by :

$$T^+(x,\xi,h,\omega) = h^{-1-\delta} \left(-\sum_{k\geq 1} 2i\sqrt{\lambda}\omega\cdot\nabla d_k\ h^{\nu_k} + (2i\xi\cdot\nabla d_k + \Delta d_k)\ h^{\nu_k+1+\delta} \right. \tag{2.15}$$

$$\left. + \sum_{k\geq 1} V_k\ h^{-(1+\delta)+\delta\rho_k} + \sum_{j,k\geq 1} V_j\ d_k\ h^{-(1+\delta)+\delta\rho_j+\nu_k} \right).$$

Roughly speaking, we construct ν_k and d_k^+ in order to obtain $T^+(h,\omega) = O(h^\infty)$.
First, we choose $\nu_1 = \delta(\rho_1 - 1) - 1$ and we solve the transport equation :

$$\omega.\nabla d_1^+(x,\omega) = \frac{1}{2i\sqrt{\lambda}} V_1(x). \tag{2.16}$$

The solution of (2.16) is given by :

$$d_1^+(x,\omega) = \frac{i}{2\sqrt{\lambda}} \int_0^{+\infty} V_1(x+t\omega)\ dt. \tag{2.17}$$

Then, we choose $\nu_2 = min\ (\nu_1 + 1 + \delta,\ -(1+\delta) + \delta\rho_1,\ 2\nu_1)$ and we solve the corresponding transport equation in order to construct d_2. The functions d_k and the coefficients ν_k for $k \geq 3$ are determined in a recursive way. □

As in [10], we have the following proposition :

Proposition 3

$$\Omega^+(h,\omega)\chi_0(h^\epsilon D)\ \Phi = \lim_{t\to+\infty} e^{itH(h,\omega)}\ J^+(h,\omega)\ e^{-itH_0(h,\omega)}\chi_0(h^\epsilon D)\ \Phi. \tag{2.18}$$

$$||\ (\Omega^+(h,\omega) - J^+(h,\omega))\ \chi_0(h^\epsilon D)\ \Phi\ || = \ O\ (h^\infty). \tag{2.19}$$

Then, Theorem 1 follows from Proposition 3 in the same way as in [10].

3 The inverse scattering problem.

Let V_j, $j = 1, 2$, be two potentials satisfying (H_2) and let $S_j(h)$, $j = 1, 2$ be the scattering operator associated with the pair $(H_0(h) + V_j, H_0(h))$.

We have the following result :

Corollary 4
For $n \geq 3$, assume that in $\mathcal{B}(L^2(I\!R^n))$,

(3.1) $$S_1(h)\chi(H_0(h)) = S_2(h)\chi(H_0(h)) + O\ (h^\infty)\ ,\ h \to 0.$$

Then : $V_1 - V_2 \in \mathcal{S}(I\!R^n)$.

Proof :
Corollary 4 follows from Theorem 1, the uniqueness for the Radon transform in hyperplanes [3], and the fact that $\| \Phi_{h,\omega} \|$, (resp. $\| \Psi_{h,\omega} \|$) are uniformly bounded with respect to h. We refer to [10] for the details. □

Comments.
It is not difficult to generalize the previous results to the case of long-range potentials, i.e for potentials satisfying (H_2) with $\rho_1 > 0$, using modified wave operators, close to Isozaki-Kitada's ones [4], (see [10], Theorem 8, for details).

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